

# Linearizability Criteria for a Class of Third Order Semi-Linear Ordinary Differential Equations

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## Abstract

Using geometric methods for linearizing systems of second order cubically semi-linear ordinary differential equations, we extend to the third order by differentiating the second order equation. This yields criteria for linearizability of a class of third order semi-linear ordinary differential equations, which is distinct from the classes available in the literature. Some examples are given and discussed.

## 1 Introduction

The study of non-linear differential equations initially centred about procedures to reduce them to linear form. The earliest attempts tried to approximate the non-linear equation by a linear one and use the solution to iteratively improve the approximation [15]. One then has to hope that the series converges. It is never very clear in this procedure in what regime the essence of non-linearity will be lost in approximation. Lie used his point transformations [8] to determine the class of scalar second order semi-linear ordinary differential equations (odes) that would transform under it to linear form [9], thus providing *exact* solutions of the odes. Such odes are called *linearizable in the sense of Lie*. More generally, one allows other transformation of variables. We shall simply call them *linearizable*. He obtained specific criteria for linearizability and obtained the most general form of the equation to be cubically semi-linear, apart from obtaining an algebraic classification of the equations that could be linearized. Some work on classification of systems of two such equations was done [17] and it was found that there are five classes of such linearizable systems. Some further work proved that for the scalar third order there are three classes

[10]. However, till recently nothing had been obtained about explicit linearizability criteria of systems of second order, or of third order odes.

Chern [2,3] used contact transformations to discuss the linearizability of scalar third order odes reducible to the linear forms  $u''' = 0$  and  $u''' + u = 0$ . Point transformations were used by Grebot [5,6] but were restricted to the class of transformations  $t = \phi(x)$ ,  $u = \psi(t, x)$ . This work was generalized by Neut and Petitot, and Ibragimov and Meleshko [16,7] to the third class known to exist [17],  $u''' + \alpha(t)u = 0$ . They obtained a larger class of third order semi-linear odes that is linearizable. Meleshko [14] also considered arbitrary point transformations to reduce third order odes of the form  $y''' = F[y, y', y'']$  and used the  $\partial/\partial t$  symmetry to reduce the order to two. He then used the Lie linearizability criteria to determine the linearizability of the scalar third order equation.

Using the connection between the isometry algebra and the symmetries of the system of geodesic equations [4], linearizability criteria were stated for a system of second order quadratically semi-linear odes, of a class that could be regarded as a system of geodesic equations [11]. The criteria came from requiring that the coefficients in the equations, regarded as Christoffel symbols, yield a zero curvature tensor. We use the projection procedure of Aminova and Aminov [1], which appeals to the fact that the geodesic equations do not depend on the geodetic parameter, to reduce the dimension by one and convert from a quadratically semi-linear system to a more general cubically non-linear system [12]. When applied to a system of two dimensions we get a single cubically semi-linear ode. The linearizability criteria so obtained are exactly the Lie criteria! Applied to a system of three dimensions, one obtains a system of two cubically semi-linear odes with extended Lie criteria. The system of odes so obtained is in the class of maximally symmetric equations (out of the total of five classes mentioned earlier). Some other classes come from projection of the original space to lower dimensional spaces.

In this paper we consider the linearizability criteria of a class of scalar third order odes obtained by differentiating the cubically semi-linear system of second order odes and stating criteria of linearizability for the class so obtained. (It is worth mentioning that we obtain the same criteria if we first differentiate the system and then project it.) The class obtained here is distinct from all the earlier classes considered as there is no guarantee that there will be three arbitrary constants appearing in the solution.

The plan of the paper is as follows. In the next section we present the geometrical notation used and briefly review the essential results used in the sequel. In section 3 we derive the linearizability criteria for the scalar third order ode in the form involving the second derivative and in a form quintic in the first derivative. In the following section we present some illustrative examples. Finally, in section 5 we give a brief summary and discussion of the results.

## 2 Notation and review

The following are well-known and can be found in text books. We use the Einstein summation convention that repeated indices are summed over the entire range of the index. Thus,  $A^a B_a$  stands for  $\sum_{a=1}^n A^a B_a$ . The metric tensor will be represented by the symmetric (non-singular) matrix  $g_{ij}$  and its inverse by  $g^{ij}$ . The Christoffel symbols are

given by

$$\Gamma_{jk}^i = \frac{1}{2}g^{im}(g_{jm,k} + g_{km,j} - g_{jk,m}), \quad (1)$$

where  $,k$  stands for partial derivative relative to  $x^k$ , etc. The Christoffel symbols are symmetric in the lower pair of indices,  $\Gamma_{jk}^i = \Gamma_{kj}^i$ .

In this notation, the system of  $n$  geodesic equations is

$$\ddot{x}^i + \Gamma_{jk}^i \dot{x}^j \dot{x}^k = 0, \quad i, j, k = 1, \dots, n, \quad (2)$$

where  $\dot{x}^i$  is the derivative relative to the arc length parameter  $s$  defined by  $ds^2 = g_{ij}dx^i dx^j$ .

The Riemann tensor is defined by

$$R_{jkl}^i = \Gamma_{jl,k}^i - \Gamma_{jk,l}^i + \Gamma_{mk}^i \Gamma_{jl}^m - \Gamma_{ml}^i \Gamma_{jk}^m, \quad (3)$$

and has the properties

$$R_{jkl}^i = -R_{jlk}^i, \quad (4)$$

$$R_{jkl}^i + R_{klj}^i + R_{ljk}^i = 0, \quad (5)$$

and

$$R_{jkl;m}^i + R_{jlm;k}^i + R_{jmk;l}^i = 0. \quad (6)$$

The Riemann tensor in fully covariant form is

$$R_{ijkl} = g_{im} R_{jkl}^m, \quad (7)$$

and satisfies

$$R_{ijkl} = -R_{jikl}. \quad (8)$$

The linearization criteria of [11] are  $R_{ijkl} = 0$ .

For a general system of two odes for two functions, if it can be regarded as a system of geodesic equations,

$$\begin{aligned} x'' &= a(x, y)x'^2 + 2b(x, y)x'y' + c(x, y)y'^2, \\ y'' &= d(x, y)x'^2 + 2e(x, y)x'y' + f(x, y)y'^2, \end{aligned} \quad (9)$$

the Christoffel symbols in terms of these coefficients are

$$\begin{aligned} \Gamma_{11}^1 &= -a, \Gamma_{12}^1 = -b, \Gamma_{22}^1 = -c, \\ \Gamma_{11}^2 &= -d, \Gamma_{12}^2 = -e, \Gamma_{22}^2 = -f. \end{aligned} \quad (10)$$

The linearizability criteria are

$$\begin{aligned} a_y - b_x + be - cd &= 0, b_y - c_x + (ac - b^2) + (bf - ce) = 0, \\ d_y - e_x - (ae - bd) - (df - e^2) &= 0, (b + f)_x = (a + e)_y, \end{aligned} \quad (11)$$

with constraints on the metric coefficients:  $g_{11} = p, g_{12} = g_{21} = q, g_{22} = r$ ,

$$\begin{aligned} p_x &= -2(ap + dq), q_x = -bp - (a + e)q - dr, r_x = -2(bq + er), \\ p_y &= -2(bp + eq), q_y = -cp - (b + f)q - er, r_y = -2(cq + fr). \end{aligned} \quad (12)$$

The compatibility of this set of six equations gives the above four linearization conditions (11). For a system of three equations we get eighteen such equations out. In general there are  $n^2(n+1)/2$ ,  $n \geq 2$ . One now obtains the required linearizing transformation by regarding the variables in which the equation is linearized as Cartesian, thus having  $g_{11} = g_{22} = 1$  and  $g_{12} = g_{21} = 0$ , and looking for the coordinate transformation yielding the original metric coefficients [11].

Following Aminova and Aminov [1] and projecting the system down by one dimension, the geodesic equations become [12]

$$\ddot{x}^a + A_{bc}\dot{x}^a\dot{x}^b\dot{x}^c + B_{bc}^a\dot{x}^b\dot{x}^c + C_b^a\dot{x}^b + D^a = 0, \quad a = 2, \dots, n, \quad (13)$$

where the dot now denotes differentiation with respect to the parameter  $x^1$  and the coefficients in terms of the  $\Gamma_{bc}^a$ s are

$$\begin{aligned} A_{bc} &= -\Gamma_{bc}^1, B_{bc}^a = \Gamma_{bc}^a - 2\delta_{(c}^a\Gamma_{b)1}^1, \\ C_b^a &= 2\Gamma_{1b}^a - \delta_b^a\Gamma_{11}^1, D^a = \Gamma_{11}^a, a, b, c = 2, \dots, n, \end{aligned} \quad (14)$$

which is cubically semi-linear. For  $n = 2$ , writing  $h = a - 2e$ ,  $g = f - 2b$ , we get the Lie criteria for the scalar equation involving the two auxiliary variables  $b$  and  $e$ , the former of which corresponds to our  $b$  and the latter to our  $-e$ :

$$\begin{aligned} 3b_x - 2g_x + h_y - 3be - 3cd &= 0, b_y - c_x + b^2 + bg + ch - ce = 0, \\ d_y + e_x - bd - dg - e^2 + eh &= 0, 3e_y - 2h_y - g_x + 3be + 3cd = 0, \end{aligned} \quad (15a)$$

which yield, in terms of the four coefficients of the cubically semi-linear equation [12],

$$\begin{aligned} 3(ch)_x + 3dc_y - 2gg_x - gh_y - 3c_{xx} - 2g_{xy} - h_{yy} &= 0, \\ 3(dg)_y + 3cd_x - 2hh_y - hg_x - 3d_{yy} - 2h_{xy} - g_{xx} &= 0. \end{aligned} \quad (15b)$$

We will use these criteria to determine the linearizability of the scalar third order semi-linear ode.

### 3 Derivation of the linearizability criteria for the third order ode

On differentiating (13) for the single differential equation, writing the independent variable as  $x$  and the dependent variable as  $y$ , we get

$$y''' + (3cy'^2 - 2gy' + h)y'' + c_yy'^4 + (c_x - g_y)y'^3 - (g_x - h_y)y'^2 + (h_x - d_y)y' - d_x = 0, \quad (16)$$

which is a total derivative. We could retain the equation in this form or use the original to remove the second order term and write the equation as quintically non-linear in the first derivative:

$$\begin{aligned} y''' - 3c^2y'^5 + (5cg + c_y)y'^4 - (4ch + 2g^2 + g_y - c_x)y'^3 \\ - (2gd + h^2 - h_x + d_y)y'^2 - (2dg + h^2 + d_y - h_x)y' + (dh - d_x) &= 0. \end{aligned} \quad (17)$$

As we show shortly, for the linearizability criteria it makes no difference which form we use. However the second, quintic, form is no longer a total derivative of the original equation except in the linear case. For purposes of comparison with the forms in the literature one needs (16) but for comparison with the previous work on the cubically semi-linear systems of odes (17) may be more convenient.

For (17) the general form of the equation is

$$y''' - \alpha y'^5 + \beta y'^4 - \gamma y'^3 + \delta y'^2 - \epsilon y' + \phi = 0. \quad (18)$$

In this case for compatibility of (17) and (18) the conditions are

$$\alpha = 3c^2, \quad (19)$$

$$\beta = 5cg + c_y, \quad (20)$$

$$\gamma = 4ch + 2g^2 + g_y - c_x, \quad (21)$$

$$\delta = 3cd + 3gh + h_y - g_x, \quad (22)$$

$$\epsilon = 2dg + h^2 + d_y - h_x, \quad (23)$$

$$\phi = dh - d_x. \quad (24)$$

On inversion, the first four equations provide definitions for the four coefficients in (13), (14) and (15b) for  $n = 2$ , and the next two provide consistency constraint conditions:

$$c = \sqrt{\alpha}/\sqrt{3}, \quad (25)$$

$$g = (\beta - c_y)/5c, (c \neq 0), \quad (26)$$

$$h = (\gamma - 2g^2 - g_y + c_x)/4c, (c \neq 0), \quad (27)$$

$$d = (\delta - 3gh - h_y + g_x)/3c, (c \neq 0), \quad (28)$$

$$\epsilon = 2dg + h^2 + d_y - h_x, (c \neq 0), \quad (29)$$

$$\phi = dh - d_x, (c \neq 0). \quad (30)$$

In the case  $c = 0$ , clearly  $\alpha = \beta = 0$ , and the equation becomes cubically non-linear. Now there can be different choices of  $g$  for a given  $\gamma$ ,  $h$  for a given  $\delta$  and choice of  $g$ , and  $d$  for a given  $\epsilon$  and choices of  $g$  and  $h$ . There is then only one consistency condition for the various choices. An example will be given in the next section to illustrate this case.

Hence we have the following theorem.

**Theorem 1:** *Equation (18) is linearizable if the linearizability criteria (15) are satisfied, where the coefficients are given by (25) - (30) ( $c \neq 0$ ), with  $h = a - 2e$ ,  $g = f - 2b$ .*

The general form corresponding to (16) is

$$y''' + (A_2 y'^2 - A_1 y' + A_0) y'' + B_4 y'^4 - B_3 y'^3 + B_2 y'^2 - B_1 y' + B_0 = 0. \quad (31)$$

The identification of most of the coefficients is *easier* in this form. We have

$$c = A_2/3, g = A_1/2, h = A_0, \quad (32)$$

but  $d$  is not obtained from here. The constraint conditions arising are:

$$B_4 = A_{2y}/3, B_3 = A_{1y}/2 - A_{1x}/3, B_2 = A_{0y} - A_{1x}/2, \quad (33)$$

which are also easier to check than the corresponding equations in the other form. There are two differential equations for  $d$  that yield its value up to an arbitrary constant

$$d = - \int B_2 dx + k(y) = \int (B_1 - A_{0x}) dy + l(x). \quad (34)$$

The constant can be determined by requiring consistency of the Christoffel symbol  $\Gamma_{11}^2$ . This is why it becomes more difficult to compute with this form of the equation. Thus we have the following theorem.

**Theorem 2:** Equation (31) is linearizable if the linearizability criteria (15) are satisfied, where the coefficients are given by (32) - (34) ( $c \neq 0$ ), with  $h = a - 2e, g = f - 2b$ , after requiring consistency of the Christoffel symbols with the deduced metric coefficients.

The four linearizability conditions (15) are stated in terms of the 6 coefficients  $a, \dots, f$  and not the four coefficients  $c, d, g, h$ . Thus there is degeneracy in the choices available. Any choices of  $a$  and  $e$  for a given  $h$ , or  $f$  and  $b$  for a given  $g$ , compatible with the metric coefficient relations (12) are permissible. For each such choice we would get corresponding linearizability conditions.

For example, assume that  $b = e = 0$ , then

$$p_x = -2(ap + dq), p_y = 0, q_x = -(aq + dr), \quad (35)$$

$$q_y = -(cp + fq), r_x = 0, r_y = -2(cq + fr). \quad (36)$$

This is a consistent set of requirements. The other four conditions become

$$h_y = cd, c_x = ch, d_y = dg, g_x = h_y. \quad (37)$$

The first two of these are

$$\begin{aligned} 4h_y &= \delta - 3gh + g_x, \\ \alpha_x/2\alpha &= [50\sqrt{3}\alpha^2\gamma + 25\alpha^{3/2} + \sqrt{3}(-2\sqrt{3}\beta + \alpha_y)^2 \\ &\quad + 5\sqrt{3}(-\alpha_y^2 + \alpha\alpha_{yy} + 2\sqrt{3}\alpha(\beta\alpha_y - \alpha\beta_y))]200\alpha^{5/2}. \end{aligned} \quad (38)$$

Hence the first and last of the four conditions are

$$g_x = h_y = \delta/3 - gh. \quad (39)$$

Since  $h$  itself depends on the second derivative of  $\alpha$  relative to  $y$ , the last condition above is a third order derivative constraint.

Alternatively, choosing  $a = f = 0$ , we would get

$$\begin{aligned} p_x &= -2dq, p_y = -2(bp + eq), q_x = -(bp + eq + dr), \\ q_y &= -(cp + bq + er), r_x = -2(bq + er), r_y = -2cq, \end{aligned} \quad (40)$$

and

$$g_x = h_y = 2cd - gh/2, 2c_x + g_y = -2(b^2 + ce), 2d_x + g_x = -(dg + h^2/2). \quad (41)$$

These are clearly more complicated and will not be used.

## 4 Examples

We present some examples of third order equations that can be linearized by our procedure.

1. Here we choose  $e = f = 0$  so that  $g = -2b, h = a$ . Further, let us try to obtain a lower degree of the equation in the form that there is no second derivative. As pointed out earlier this requires that  $c = 0$ . To construct this example we choose the metric coefficients  $p = A(x)y^2, q = B'(x)y, r = 2B(x)$ . Writing  $pr - q^2 = \Delta y^2$  and  $A^2 + A'B'/2 - AB'' = \Lambda$ , we get  $h = a = -(\ln\sqrt{\Delta})', g = -2b = 2/y, c = 0, d = \Lambda y/\Delta, e = f = 0$ , where the prime here refers to differentiation with respect to only  $x$  and *not* relative to both  $x$  and  $y$ . Notice that the degeneracy of the case when  $c = 0$  has been taken care of by selecting the metric coefficients appropriately. Here, choosing  $A = c_1 e^{-kx}, B = c_2 e^{-kx}$  the equation becomes

$$y''' - 6y'^3/y^2 + 8ky'^2/y - (k^2 - 5l)y' + kly = 0, \quad (42)$$

where  $c_1 = 2lc_2$ . This is amenable to reduction by the method of [14] and is not of a form given in [16] or [7].

2. Here we construct the example by differentiating the linearizable equation

$$y'' + xy'^3 + \frac{2}{x}y' = 0, \quad (43)$$

which yields

$$y''' - 3x^2y'^5 - 7y'^3 - \frac{6}{x^2}y' = 0. \quad (44)$$

This has  $c = x, d = 0, g = f - 2b = 0$  and  $h = a - 2e = 2/x$ . We choose  $b = 0$  and  $e = -1/x$ . Then  $f = 0$  and  $a = 0$ . It is easily verified that the conditions are met and that it is not in any of the classes of [16], [7] or [14] and cannot be linearized according to the methods therein. It can easily be reduced to the simplest linear form by the transformation  $u = x \cos y, v = x \sin y$ . Here the solution is in terms of two arbitrary constants.

3. In this example the equation taken is

$$y''' - 3x^2y'^5/y^4 - 3xy'^4/y^3 + 6y'^3/y^2 + 6y'^2/xy - 6y'/x^2 = 0. \quad (45)$$

Since the equation is non-trivial in that it is not in the class of Neut and Petiot [16], Ibragimov and Meleshko [7] or Meleshko [14], is not a total derivative, nor can it be solved by other simple means, we give the steps in some detail. From (25) - (30), we see that  $c = -x/y^2, d = 0, h = 2/x, g = 1/y$ . Choosing  $a = b = 0$  we have  $f = g, e = -h/2$ . The metric coefficients are then given by (12) to be

$$p = y^2(1 + y^{-4})/2, q = xy(1 - y^{-4})/2, r = x^2(1 + y^{-4})/2. \quad (46)$$

Writing the Cartesian coordinates as  $(u, v)$ , the coordinate transformations are given by ([11], eqs. (32) - (34))

$$u_x^2 + v_x^2 = y^2 + y^{-2}, u_x u_y + v_x v_y = x(y - y^{-3}), u_y^2 + v_y^2 = x^2(y^2 + y^{-2}). \quad (47)$$

This set of equations is easily solved by setting

$$u_x = y, u_y = x, v_x = y^{-1}, v_y = -xy^{-2}, \quad (48)$$

which yields

$$u = xy, v = xy^{-1}, \quad (49)$$

which are the linearizing transformations. Thus the solution is

$$Axy + Bx/y = 1, \quad (50)$$

where  $A$  and  $B$  are arbitrary constant real numbers. Notice that this equation is quintic in the first derivative and does depend on  $x, y, y'$ , which is not in the classes of [16], [7] or [14].

## 5 Concluding remarks

We have shown that the criteria for linearizability of a third order semi-linear ode that were available in the literature [16, 7] exclude classes of third order equations that can be obtained by differentiating second order linearizable equations. By differentiating the general second order cubically semi-linear ode we can check the linearizability of the class of third order semi-linear equations that are products of a quadratic factor in the first order times the second order term and quartic in the first order terms. The class can be more conveniently expressed as a quintic in the first order. In this case it is no longer a total derivative of the original second-order equation (except in the linear case) so that it is now hidden that this third order equation arises from a second order one. As such, our procedure gives a *non-classical solution*. The criteria are explicit for the quintic form but become degenerate in the case that the factor reduces to a cubic. It is worth noting that this procedure does not allow for a quartic factor only. Some examples were given.

The procedure presented here leaves freedom of choice of the coefficients of the system of quadratic equations that yields the cubic. The reason is that there were six coefficients of the two-dimensional quadratic system, while only four combinations of them enter into the third order equation. There are, indeed, six coefficients in the general third order equation of the desired type. The extra two coefficients yield additional constraint equations for the compatibility of the cubic form with the derivative of the quadratic form. We then only have to check that the Lie criteria are satisfied by the corresponding second order equation.

Since [16] and [7] used the classic Lie procedure and obtained all the classes possible for the third order ode to be linearizable by point transformations, the question arises how we have got additional classes. The answer is that we are not using point transformations *directly*. We use point transformations and then differentiate. This could be something like contact transformations (as the derivative is also involved) but it is not quite a contact transformation procedure either. Note that our procedure does not guarantee that there will be three arbitrary constants. We only guarantee two. The procedure of [16] does guarantee three. It would be interesting if a procedure could be found that involves only a single constant.

Our procedure is also interesting in that it can yield criteria for systems of third order odes. Work is in progress for a system of third order cubically semi-linear system [13]. The equivalent of the present work for a system is in principle possible but appears too messy at present. Better algebraic computing software may make it more manageable.



Another line of work that will be useful is to find methods for the degeneracy in our procedure to be removed. Alternatively, we need to determine all classes that allow for the degeneracy, where it cannot be removed.

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